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DYNAMICS OF THE LONGITUDINAL PROPAGATION OF ELASTIC DISTURBANCE THROUGH A MEDIUM EXHIBITING GRADIENT OF ELASTICITY

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ABSTRACT. Extensional vibration in an isotropic medium for linear variation of elastic parameters has been considered. The problem is worked out following Operational methods. Two distinct cases have been worked from the general solution, namely, (1) for a source having impulsive force at one end, the other end remaining free, and (2), impulsive force at one end, the other end being fixed. Solution obtained in the form of modified Bessel functions have further been simplified for small variations of the parameter using method of steepest descents as adopted by Debye.

INTRODUCTION

The general problem of the extensional vibration of a bar excited by the impact of an elastic load has already been solved for a number of cases by the author, following Operational method. The theory has been further extended to include dynamics of plastic deformations in a bar exhibiting strain-rate effect and subjected to (1) impact stress, (2) alternating stress.

In the present paper, an isotropic elastic medium of uniform density ρ is considered, where the elastic parameters λ, μ, are supposed to vary linearly.

Explanations of the symbols used:

l = Length of the medium

t = Variable time

x = Variable length, measured in the direction of propagation of the disturbance, the medium being free at $x = 0$ and impacted at $x = l$ (as in sec. I). But in section II, the medium is supposed fixed at $x = 0$ and impacted at $x = l$.

U = Displacement at any section.

U_l = Displacement at $x = l$.

ρ = Density of the medium (supposed uniform)

λ, μ = Elastic parameters whose linear variations are supposed in accordance with the relations,

$$\left.\begin{aligned}\lambda &= \lambda_0+\lambda_1 x\\ \mu &= \mu_0+\mu_1 x\end{aligned}\right\}\quad (\lambda_1, \mu_1 > 0)$$

where λ_0, μ_0 are the values at the origin, $x = 0$

v_0 = Velocity of impact.

a_0 = Compressional wave velocity at $x = 0$

J = Impulse per unit area.

P = Pressure of impact.

D = Operator d/dt.

The differential equation governing motion in one dimension in an isotropic elastic medium of uniform density ρ is given by

$$\frac{d}{dx}\left[(\lambda + 2\mu)\frac{du}{dx}\right] = \rho\frac{d^2u}{dt^2} \qquad \ldots (1)$$

Using transformation

$$Z = \lambda_0 + 2\mu_0 + (\lambda_1 + 2\mu_1)x \qquad \ldots (2)$$

Eq. (1) reduces to

$$Z\frac{d^2u}{dz^2} + \frac{du}{dz} = C^2\frac{d^2u}{dt^2} \qquad \ldots (1.1)$$

which is equivalent to

$$Z\frac{d^2u}{dz^2} + \frac{du}{dz} - C^2D^2u = 0 \qquad \ldots (1.2)$$

where $$C^2 = \rho/(\lambda_1 + 2\mu_1)^2. \qquad \ldots (3)$$

The substitution

$$y = 2CD[\lambda_0 + 2\mu_0 + (\lambda_1 + 2\mu_1)x]^{\frac{1}{2}} \qquad \ldots (4)$$

reduces Eq. (1.2) to

$$\frac{d^2u}{dy^2} + \frac{1}{y}\frac{du}{dy} - u = 0 \qquad \ldots (1.3)$$

which is modified Bessel's equation of Zero order and has the solution,

$$U(x, t) = AI_0(y) + BK_0(y) \qquad \ldots (5)$$

where I_0, K_0 are modified Bessel Functions of zero order.

For large values of y (since C is large for small values of λ_1, μ_1), Eq. (5) can be approximately written, using the method of steepest descents, as adopted by Debye, as

$$U(x, t) = y^{-\frac{1}{2}}[A_1e^{y} + B_1e^{-y}] \qquad \ldots (6)$$

SECTION I.

The terminal conditions are :

at the free end $x = 0$, $$\frac{du}{dx} = 0 \quad \ldots \quad (6.1)$$

and at the end $x = l$, $$u = U_l \quad \ldots \quad (6.2)$$

Conditions (6.1) and (6.2) reduce equation (6) to

$$U(x, t) = (Z/Z_l)^{-1/4} \cdot \frac{\cosh 2CD(Z^{\frac{1}{2}} - Z_0^{\frac{1}{2}})}{\cosh 2CD(Z_l^{\frac{1}{2}} - Z_0^{\frac{1}{2}})} \cdot U_l \quad \ldots \quad (7)$$

where
$$\left.\begin{aligned} Z_l &= \lambda_0 + 2\mu_0 + (\lambda_1 + 2\mu_1)l \\ Z_0 &= \lambda_0 + 2\mu_0 \end{aligned}\right\} \quad \ldots \quad (7.1)$$
and

Now the pressure of impact on the medium at $x = l$ is given by

$$P = -\left(Z\frac{du}{dx} \right)_{x=l} \quad \ldots \quad (8)$$

An impulse J per unit area is given to the medium, at $x = l$, and the subsequent equation of motion is given by

$$J/v_0 \cdot \frac{d^2U_l}{dt^2} = -\left(Z\frac{du}{dx} \right)_{x=l} \quad \ldots \quad (9)$$

Now substituting the value of $\left(Z\frac{du}{dx} \right)_{x=l}$ as obtained from Eq. (7) in Eq. (9) and imposing the boundary conditions, we have

$$D\rho a_0 \left[1 + \frac{1}{2}\frac{(\lambda_1 + 2\mu_1)l}{\lambda_0 + 2\mu_0} \right] U_l \tanh \frac{Dl}{a_0} + J/v_0\, D^2U_l = JD. \quad \ldots \quad (10)$$

retaining up to first power of λ_1, μ_1 and using the condition

$$l < (\lambda_0 + 2\mu_0)/(\lambda_1 + 2\mu_1)$$

Eq. (10) yields,

$$U_l = v_0/F(D). \quad \ldots \quad (11)$$

where $F(D)$

$$= D + \frac{\rho a_0 v_0}{J} \left\{ 1 + \tfrac{1}{2} \frac{(\lambda_1 + 2\mu_1)l}{\lambda_0 + 2\mu_0} \right\} \tanh \frac{Dl}{a_0}$$

$$= D + q . \frac{1 - e^{-2Dl/a_0}}{1 + e^{-2Dl/a_0}} \qquad \ldots (12)$$

where

$$q = \frac{\rho a_0 v_0}{J} \left\{ 1 + \tfrac{1}{2} \frac{(\lambda_1 + 2\mu_1)l}{\lambda_0 + 2\mu_0} \right\} \qquad .. (12.1)$$

Eq. (11) with the help of (12) becomes

$$U_l = \Bigg[\frac{1}{D+q} + \left\{ \frac{1}{D+q} - \frac{D-q}{(D+q)^2} \right\} e^{-2Dl/a_0}$$

$$\left\{ \frac{(D-q)^2}{(D+q)^3} - \frac{D-q}{(D+q)^2} \right\} e^{-4Dl/a_0} + \ldots + \ldots \Bigg] v_0$$

$$= [f_1(t) + 2f_1(t_1) - 2f_2(t_1), + 4f_3(t_2) - 6f_2(t_2) + 2f_1(t_1), + \ldots + \ldots] \qquad \ldots (13)$$

where $f(t_n)$ denotes $f(t - n\theta_1) = f(t - n.2l/a_0)$.

The values of the functions are similar to those obtained by the author earlier (Ghosh and Ghosh, 1951).

DISPLACEMENT AT ANY SECTION

Eq. (7) when expanded in terms of its equivalent exponential and simplified using small values of λ_1, μ_1 up to first power gives with the help of Eq. (13),

$$U(x, t) = \left\{ 1 + \frac{1}{4} \frac{(\lambda_1 + 2\mu_1)(l-x)}{\lambda_0 + 2\lambda_0} \right\} \sum_{r=1}^{n} \Big[e^{D/a_0\{x - l - r - 1.2l\}}$$

$$+ e^{-D/a_0\{x - l + r.2l\}} \Big] U_l \qquad \ldots (14)$$

where r is an integer.

Eq. (14) is the general form, giving the displacement at any section at any instant during impact.

Now substituting the value of U_l from Eq. (13) in Eq. (14) and collecting only the useful terms occuring during the desired interval of time, we get for the displacement at any section during $0 < t < 2l/a_0$,

$$U(x, t) = \left\{ 1 + \frac{1}{4} \frac{(\lambda_1 + 2\mu_1)(l-x)}{\lambda + 2\mu_0} \right\} \left[e^{\frac{D}{a_0}(x-l)} + e^{-\frac{D}{a_0}(x+l)} \right] f_1(t) \ldots (14.1)$$

$$= \frac{J}{\rho a_0} \left[1 - \frac{1}{4} \frac{(\lambda_1+2\mu_1)(x+l)}{\lambda_0+2\mu_0} \right] \left[1 - e^{-\frac{q}{a_0}(a_0 t - l - x)} \right] \quad \ldots \quad (14.2)$$

Since $a_0 \times v_0$ is large (when v_0 is large), q is large and at $t = l/a_0$ equation (14.2) reduces to

$$U(x, t) = \frac{Jt}{\rho l} \left[1 - \frac{1}{4} \frac{(\lambda_1+2\mu_1)(l+x)}{\lambda_0+2\mu_0} \right] \quad \ldots \quad (14.3)$$

Eq. (14.3) is similar to that obtained by A. N. Dutta (1956) and is a particular case derived from the general solution given by Eq. (14.2).

It is clear that Eq. (14.3) fails to give general displacement $U(x, t)$ for time $t < l/a_0$ *i.e.*, until the waves have reached the far end. Further, the displacement Eq. (13) shows that the wave train does not return after reflection, as shown by the second term of Eq. (15) below.

PRESSURE AT THE IMPACTED END

Combining Eq. (8) with (7), the pressure of impact on the medium at $x = l$, *i.e.* impacted end is numerically given by

$$P = \rho a_0 \left[1 + \frac{1}{2} \frac{(\lambda_1+2\mu_1)l}{\lambda_0+2\mu_0} \right] \tanh \frac{Dl}{a_0} \cdot U'_l$$

$$= \rho a_0 \left[1 + \frac{1}{2} \frac{(\lambda_1+2\mu_1)l}{\lambda_0+2\mu_0} \right] \left[f_1'(t) - 2f_2'(t_1) + 4f_3'(t_2) - 2f_2'(t_2) + \ldots \right] \quad \ldots \quad (15)$$

Thus during $0 < t < 2l/a_0$,

$$P_1 = \rho a_0 \left[1 + \frac{1}{2} \frac{(\lambda_1+2\mu_1)l}{\lambda_0+2\mu_0} \right] \cdot v_0 e^{-qt}. \quad \ldots \quad (15.1)$$

SECTION II

The terminal conditions are:

at the fixed end, $x = 0$, $U = 0$ ... (16.1)

and at the end, $x = l$, $U = U_l$ ... (16.2)

Conditions (16.1) and (16.2) reduce equation (6) to

$$U(x, t) = \left(\frac{Z}{Z_l} \right)^{-1/4} \cdot \frac{\sinh 2CD(Z^{\frac{1}{2}} - Z_0^{\frac{1}{2}})}{\sinh 2CD(Z_l^{\frac{1}{2}} - Z_0^{\frac{1}{2}})} \cdot U_l \quad \ldots \quad (17)$$

Now substituting the value of $\left(Z\frac{du}{dx}\right)_{x=l}$ from Eq. (17) in Eq. (9) and imposing the boundary conditions we have,

$$D\rho a_0\left[1+\frac{1}{\lambda}\frac{(\lambda_1+2\mu_1)l}{\lambda_0+2\mu_0}\right]U_l \coth\frac{Dl}{a_0}+J/v_0\, D^2U_l = JD \qquad \ldots \quad (18)$$

retaining up to first power of λ_1, μ_1 and subject to the condition that $l < \frac{\lambda_0+2\mu_0}{\lambda_1+2\mu_1}$

Eq. (18) yields

$$U_l = \frac{v_0}{F(D)} \qquad (19)$$

where
$$F(D) = D+\frac{\rho a_0 v_0}{J}\left\{1+\frac{1}{2}\frac{(\lambda_1+2\mu_1)l}{\lambda_0+2\mu_0}\right\}\coth\frac{Dl}{a_0}$$

$$= D+q \coth \frac{Dl}{} \qquad (20)$$

where
$$q = \frac{\rho a_0 v_0}{J}\left\{1+\frac{1}{2}\frac{(\lambda_1+2\mu_1)l}{\lambda_0+2\mu_0}\right\} \text{ as before.}$$

Eq. (19) with the help of Eq.(20) becomes

$$U_l = f_1(t),+2f_2(t_1)-2f_1(t_1),+4f_3(t_2)-6f_2(t_2)+2f_1(t_2),$$

$$+\ldots++2\left[2^{n-1}f_{n+1}(t_n)-2^{n-2}.\frac{n+1}{n}\,{}^nC_1f_n(t_n)+2^{n-3}.\frac{n+2}{n}\,{}^nC_2f_{n-1}(t_n)\right.$$

$$\left.-+\ldots+(-1)^nf_1(t_n)\right].$$

The values of the functions are the same as those in Section I.

REFERENCES

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